BSC (Math) part I

paper – II

Topic :- The shortest distance (3D Geometry)

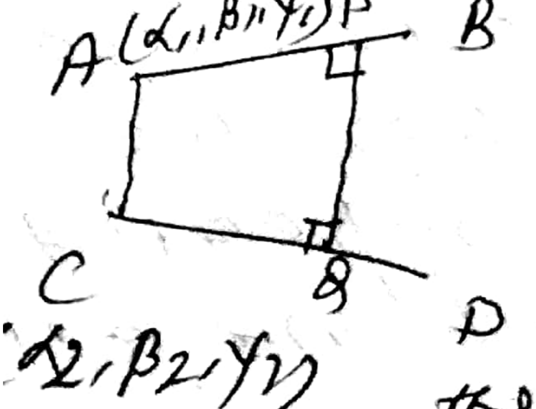
Theorem :- To find shortest distance between the straight lines $\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1}$

and $\frac{x-\alpha_1}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$

Proof :- Let AB and CD be the two skew lines whose eqn. are

$$\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \quad \text{--- (1)}$$

and $$\frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} \quad \text{--- (2)}$$



Let the coordinates of A and C be $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ respectively.

Let PQ be the shortest distance between the lines AB and CD

Then PQ is perpendicular to both the lines

let the direction cosine of PQ be l, m, n then we have

$ll_1 + mm_1 + nn_1 = 0$

and $ll_2 + mm_2 + nn_2 = 0$

By cross multiplication we have

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} = r \text{ (say)}$$

$\therefore l = r(m_1n_2 - n_2m_1)$

$m = r(n_1l_2 - n_2l_1)$

and $n = r(l_1m_2 - l_2m_1)$

By squaring and adding we get

$$l^2 + m^2 + n^2 = r^2\{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2\}$$

~~If θ be angle between the lines then~~

~~cos θ =~~

$\therefore 1 = r^2 \sum (m_1 n_2 - m_2 n_1)$

$\therefore r = \dfrac{1}{\sqrt{\sum m_1 n_2 - m_2 n_1}}$

The projection of AC on the line PQ is equal to PQ

$\therefore (\alpha_1 - \alpha_2)^2 l + (\beta_1 - \beta_2) m + (\gamma_1 - \gamma_2) n = PQ$

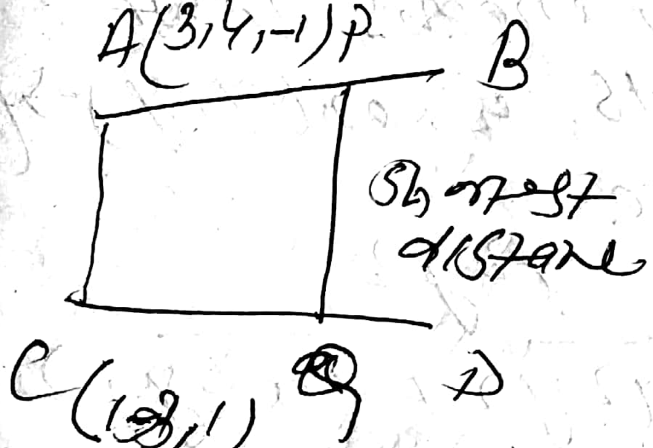
& $\sum [(\alpha_1 - \alpha_2)(m_1 n_2 - m_2 n_1) + (\beta_1 - \beta_2)(n_1 l_2 - n_2 l_1) + (\gamma_1 - \gamma_2)(l_1 m_2 - l_2 m_1)] r = PQ$

& $PQ = \begin{vmatrix} \alpha_1 - \alpha_2 & \beta_1 - \beta_2 & \gamma_1 - \gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\sum m_1 n_2 - m_2 n_2}$

This is required length of shortest distance.

problem :- Find the length of shortest distance between lines $\frac{x-3}{2} = \frac{y-4}{1} = \frac{z+1}{-3}$

and $\frac{x-1}{-1} = \frac{y-3}{3} = \frac{z-1}{2}$



Soln :- Let eqn of AB and CD be $\frac{x-3}{2} = \frac{y-4}{1} = \frac{z+1}{-3}$

and $\frac{x-1}{-1} = \frac{y-3}{3} = \frac{z-1}{2}$

Then Co-ordinates of A and C are $(3,4,-1)$ and $(1,3,1)$

Let (l, m, n) be the direction cosines of the shortest distance PQ between AB and CD.

Then by the definition of shortest distance PQ is perp. to both AB and CD

$\therefore \; 2l + m - 3n = 0$

and $\; -l + 3m + 2n = 0$ (By formula $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$)

By cross multiplication we have

$$\frac{l}{2+9} = \frac{m}{3-4} = \frac{n}{6+1}$$

& $\frac{l}{11} = \frac{m}{-1} = \frac{n}{7} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{121+1+49}} = \frac{1}{\sqrt{171}}$

$\therefore l = \frac{11}{\sqrt{171}}, \; m = -\frac{1}{\sqrt{171}}, \; n = \frac{7}{\sqrt{171}}$

The projection of AC on the line PQ is equal to PQ

$\therefore PQ = (x_2 - x_1) l + (y_2 - y_1) m + (z_2 - z_1) n$

$= (3-1)\frac{11}{\sqrt{171}} + (4-3)\frac{-1}{\sqrt{171}} + (-1-1)\frac{7}{\sqrt{171}}$

$= \frac{7}{\sqrt{171}}$ (numerically)